

# Strongly singular MASAs and mixing actions in finite von Neumann algebras

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*Abstract.* Let  $\Gamma$  be a countable group and let  $\Gamma_0$  be an infinite abelian subgroup of  $\Gamma$ . We prove that if the pair  $(\Gamma, \Gamma_0)$  satisfies some combinatorial condition called (SS), then the abelian subalgebra  $A = L(\Gamma_0)$  is a singular MASA in  $M = L(\Gamma)$  which satisfies a weakly mixing condition. If, moreover, it satisfies a stronger condition called (ST), then it provides a singular MASA with a strictly stronger mixing property. We describe families of examples of both types coming from free products, Higman–Neumann–Neumann extensions and semidirect products, and in particular we exhibit examples of singular MASAs that satisfy the weak mixing condition but not the strong mixing one.

## 1. Introduction

If  $M$  is a von Neumann algebra, and if  $A$  is a maximal abelian von Neumann subalgebra of  $M$  (henceforth called a MASA), let  $\mathcal{N}_M(A)$  be the *normalizer* of  $A$  in  $M$ : it is the subgroup of the unitary group  $U(M)$  of all elements  $u$  such that  $uAu^* = A$ . Then  $A$  is *singular* in  $M$  if  $\mathcal{N}_M(A)$  is as small as possible—that is, if  $\mathcal{N}_M(A) = U(A)$ . Until recently, it was quite difficult in general to exhibit singular MASAs in von Neumann algebras, though Popa [11] among others proved that all separable type II<sub>1</sub> factors admit singular MASAs.

*Example.* This example is due to Radulescu [12]. Let  $L(F_N)$  be the factor associated with the non-abelian free group on  $N$  generators  $X_1, \dots, X_N$  and let  $A$  be the abelian von Neumann subalgebra generated by  $X_1 + \dots + X_N + X_1^{-1} + \dots + X_N^{-1}$ . Then  $A$  is a singular MASA in  $L(F_N)$ .  $A$  is called the *radial* or *Laplacian* subalgebra because its elements coincide with convolution operators by functions that depend only on the length of the elements of  $F_N$ .

Recently, Bildea [2] generalized Radulescu's example by using the notion of asymptotic homomorphism (see below): Let  $G$  be either  $F_N$  or a free product  $G_1 \star \cdots \star G_m$  of  $m \geq 3$  groups, each finite of order  $p \geq 2$  with  $m \geq p$ . Then the radial subalgebra is a singular MASA in  $L(G)^{\otimes k}$ .

Motivated by Popa's articles [10, 11], the authors of [14] and [17] introduced sufficient conditions on an abelian von Neumann subalgebra  $A$  of a finite von Neumann algebra  $M$  that imply that  $A$  is even a strongly singular MASA in  $M$ . This means that  $A$  satisfies the apparently stronger condition that, for all  $u \in U(M)$ ,

$$\sup_{x \in M, \|x\| \leq 1} \|E_A(x) - E_{uAu^*}(x)\|_2 \geq \|u - E_A(u)\|_2.$$

In fact, it was proved by Sinclair *et al* [19] that all singular MASAs are strongly singular. Nevertheless, it is sometimes easier to prove strong singularity directly.

**PROPOSITION 1.1.** [14] *Suppose that the pair  $A \subset M$  satisfies the following condition (WM): for all  $x, y \in M$  and for all  $\varepsilon > 0$ , there exists  $v \in U(A)$  such that*

$$\|E_A(xvy) - E_A(x)vE_A(y)\|_2 \leq \varepsilon.$$

*Then  $A$  is a strongly singular MASA in  $M$ .*

Let us reproduce an adaptation of the proof of [14, Lemma 2.1] for convenience.

*Proof.* Fix  $u \in U(M)$  and  $\varepsilon > 0$ , and take  $x = u^*$ ,  $y = u$ . There exists  $v \in U(A)$  such that

$$\|E_A(u^*vu) - E_A(u^*)vE_A(u)\|_2 = \|E_A(v^*u^*vu) - E_A(u^*)E_A(u)\|_2 \leq \varepsilon.$$

(Commutativity of  $A$  is crucial here.) Hence, we get:

$$\begin{aligned} \|E_A - E_{uAu^*}\|_{\infty,2}^2 &\geq \|v - uE_A(u^*vu)u^*\|_2^2 \\ &= \|u^*vu - E_A(u^*vu)\|_2^2 \\ &= 1 - \|E_A(u^*vu)\|_2^2 \\ &\geq 1 - (\|E_A(u^*)vE_A(u)\|_2 + \varepsilon)^2 \\ &\geq 1 - (\|E_A(u)\|_2 + \varepsilon)^2 \\ &= \|u - E_A(u)\|_2^2 - 2\varepsilon\|E_A(u)\|_2 - \varepsilon^2. \end{aligned}$$

As  $\varepsilon$  is arbitrary, we get the desired inequality.  $\square$

Earlier, Sinclair and Smith [17] used a stronger condition (which we call *condition (AH)* here) in order to get singular MASAs.

Given  $v \in U(A)$ , the conditional expectation  $E_A$  is an *asymptotic homomorphism with respect to  $v$*  if

$$(AH) \quad \lim_{|k| \rightarrow \infty} \|E_A(xv^k y) - E_A(x)v^k E_A(y)\|_2 = 0$$

for all  $x, y \in M$ .

Both conditions (WM) and (AH) remind one of mixing properties of group actions, because of the following equality (since  $A$  is abelian):

$$\|E_A(vxv^*y) - E_A(x)E_A(y)\|_2 = \|E_A(xv^*y) - E_A(x)v^*E_A(y)\|_2$$

for all  $x, y \in M$ , for all  $v \in U(A)$ .

Thus our point of view here is the following: every subgroup  $G$  of  $U(A)$  acts by inner automorphisms on  $M$  by conjugation

$$\sigma_v(x) = vxv^* \quad \text{for all } v \in G, \text{ for all } x \in M,$$

and as will be seen, condition (WM) is in some sense equivalent to a weakly mixing action of  $G$ , and condition (AH) is equivalent to a strongly mixing action of the cyclic subgroup of  $U(A)$  generated by the distinguished unitary  $v$ .

Section 2 is devoted to weakly mixing actions of subgroups  $G \subset U(A)$ , to crossed products by weakly mixing actions of countable groups, and to pairs  $(\Gamma, \Gamma_0)$  where  $\Gamma_0$  is an abelian subgroup of the countable group  $\Gamma$  which provide pairs  $A = L(\Gamma_0) \subset M = L(\Gamma)$  with  $A$  weakly mixing in  $M$  (see Definition 2.1). It turns out that the weak mixing property is completely determined by a combinatorial property of the pair  $(\Gamma, \Gamma_0)$  called *condition (SS)*, already introduced in [14] to provide examples of strongly singular MASAs.

Section 3 is devoted to a (strictly) stronger condition (called *condition (ST)*) on pairs  $(\Gamma, \Gamma_0)$  as above which is related to strongly mixing actions of groups.

Section 4 contains various families of examples (free products with amalgamation, Higman–Neumann–Neumann (HNN) extensions, semidirect products), and some of them prove that condition (ST) is strictly stronger than condition (SS) so that they provide two distinct levels of ‘mixing MASAs’: weak ones and strong ones.

We will see that many pairs  $(\Gamma, \Gamma_0)$  satisfy a strictly stronger condition than condition (ST), based on malnormal subgroups:  $\Gamma_0$  is said to be a *malnormal subgroup* of  $\Gamma$  if, for every  $g \in \Gamma \setminus \Gamma_0$ ,  $g\Gamma_0g^{-1} \cap \Gamma_0 = \{1\}$ . Such pairs were first considered in the pioneering article [10] in order to control normalizers (in particular, relative commutants) of  $L(\Gamma_0)$  and of its diffuse subalgebras, and, as a byproduct, to produce singular MASAs  $L(\Gamma_0)$ . They were also used in [13, 17] to provide more examples of (strongly) singular MASAs in type  $\text{II}_1$  factors that fit Popa’s criteria of [10, Proposition 4.1].

## 2. Weak mixing

In the rest of this paper,  $M$  denotes a finite von Neumann algebra, and  $\tau$  is some normal, faithful, finite, normalized trace on  $M$  (henceforth simply called a *trace* on  $M$ ). It defines a scalar product on  $M$ ,  $\langle a, b \rangle = \tau(b^*a) = \tau(ab^*)$ , and the corresponding completion is the Hilbert space  $L^2(M, \tau)$  on which  $M$  acts by left multiplication extending the analogous operation on  $M$ . As usual, we denote by  $\|\cdot\|_2$  the corresponding Hilbert norm. When the trace  $\tau$  must be specified, we write  $\|\cdot\|_{2,\tau}$ . We also denote by  $M_\star$  the predual of  $M$ , i.e. the Banach space of all normal linear functionals on  $M$ . We will always assume for convenience that  $M_\star$  is separable, or equivalently, that  $L^2(M, \tau)$  is a separable Hilbert space. Recall that, for every  $\varphi \in M_\star$ , there exist  $\xi, \eta \in L^2(M, \tau)$  such that  $\varphi(x) = \langle x\xi, \eta \rangle$  for all  $x \in M$ .

Let  $\Gamma$  be a countable group and let  $\Gamma_0$  be an abelian subgroup of  $\Gamma$ . Denote by  $L(\Gamma)$  ( $L(\Gamma_0)$ ) the von Neumann algebra generated by the left regular representation  $\lambda$  of  $\Gamma$  ( $\Gamma_0$ ). Recall that  $\lambda : \Gamma \rightarrow U(\ell^2(\Gamma))$  is defined by  $(\lambda(g)\xi)(h) = \xi(g^{-1}h)$  for all  $g, h \in \Gamma$  and  $\xi \in \ell^2(\Gamma)$ . This extends linearly to the group algebra  $\mathbb{C}\Gamma$ , and  $L(\Gamma)$  is the weak-operator closure of  $\lambda(\mathbb{C}\Gamma) =: L_f(\Gamma)$ . The normal functional  $\tau(x) = \langle x\delta_1, \delta_1 \rangle$  is a faithful trace on  $L(\Gamma)$ . For  $x \in L(\Gamma)$ , denote by  $\sum_{g \in \Gamma} x(g)\lambda(g)$  its ‘Fourier expansion’:

$x(g) = \tau(x\lambda(g^{-1}))$  for every  $g \in \Gamma$ , and the series  $\sum_g x(g)\lambda(g)$  converges to  $x$  in the  $\|\cdot\|_2$  sense so that  $\sum_{g \in \Gamma} |x(g)|^2 = \|x\|_2^2$ .

Let  $1 \in B$  be a unital von Neumann subalgebra of the von Neumann algebra  $M$  endowed with some trace  $\tau$  as above and let  $E_B$  be the  $\tau$ -preserving conditional expectation of  $M$  onto  $B$ . Then  $E_B$  is characterized by two conditions:  $E_B(x) \in B$  for all  $x \in M$ , and  $\tau(E_B(x)b) = \tau(xb)$  for all  $x \in M$  and all  $b \in B$ . It enjoys the following well-known properties:

- (1)  $E_B(b_1xb_2) = b_1E_B(x)b_2$  for all  $x \in M$  and all  $b_1, b_2 \in B$ ;
- (2)  $\tau \circ E_B = \tau$ ;
- (3) if  $M = L(\Gamma)$  is the von Neumann algebra associated with the countable group  $\Gamma$ , if  $H$  is a subgroup of  $\Gamma$ , if  $B = L(H)$  and if  $x \in M$ , then  $E_B(x) = \sum_{h \in H} x(h)\lambda(h)$ .

Let also  $\alpha$  be a  $\tau$ -preserving action of  $\Gamma$  on  $M$ . Recall that it is *weakly mixing* if, for every finite set  $F \subset M$  and for every  $\varepsilon > 0$ , there exists  $g \in \Gamma$  such that

$$|\tau(\alpha_g(a)b) - \tau(a)\tau(b)| < \varepsilon \quad \text{for all } a, b \in F.$$

Popa [8] introduced a relative version of weakly mixing actions: If  $1 \in A \subset M$  is a von Neumann subalgebra such that  $\alpha_g(A) = A$  for all  $g \in \Gamma$ , the action  $\alpha$  is called *weakly mixing relative to A* if, for every finite set  $F \subset M \ominus A := \{x \in M : E_A(x) = 0\}$ , for every  $\varepsilon > 0$ , one can find  $g \in \Gamma$  such that

$$\|E_A(x^*\alpha_g(y))\|_2 \leq \varepsilon \quad \text{for all } x, y \in F.$$

Here we consider a (countable) subgroup  $G$  of the unitary group  $U(A)$  which acts on  $M$  by conjugation:

$$\sigma_v(x) = vxv^* \quad \text{for all } v \in G, \text{ for all } x \in M.$$

This allows us to introduce the following definition.

**Definition 2.1.** The abelian von Neumann subalgebra  $A$  is *weakly mixing in M* if there exists a subgroup  $G$  of  $U(A)$  such that the corresponding action by conjugation is weakly mixing relative to  $A$  in Popa's sense.

Notice that,  $A$  being abelian, this is equivalent to requiring that, for every finite set  $F \subset M$  and every  $\varepsilon > 0$ , there exists  $v \in G$  such that

$$\|E_A(xvy) - E_A(x)vE_A(y)\|_2 \leq \varepsilon \quad \text{for all } x, y \in F.$$

Weakly mixing actions provide singular MASAs, as was already widely known [7, 9, 14]. More precisely, we can prove the following proposition.

**PROPOSITION 2.2.** Let  $\Gamma_0$  be an abelian group which acts on a finite von Neumann algebra  $B$  and which preserves a trace  $\tau$ . Then the abelian von Neumann subalgebra  $A = L(\Gamma_0)$  of the crossed product  $M = B \rtimes \Gamma_0$  is weakly mixing in  $M$  if and only if the action of  $\Gamma_0$  is.

*Sketch of proof.* Observe first that, for  $x = \sum_\gamma x(\gamma)\lambda(\gamma) \in M$ , the projection of  $x$  onto  $A$  is given by  $E_A(x) = \sum_\gamma \tau(x(\gamma))\lambda(\gamma)$ .

Suppose that the action  $\alpha$  of  $\Gamma_0$  is weakly mixing. It suffices to prove that, for all finite sets  $E \subset \Gamma_0$  and  $F \subset B$  and for every  $\varepsilon > 0$ , there exists  $\gamma \in \Gamma_0$  such that

$$\|E_A(a\lambda(g)\lambda(\gamma)b\lambda(h)) - E_A(a\lambda(g))\lambda(\gamma)E_A(b\lambda(h))\|_2 < \varepsilon$$

for all  $a, b \in F$  and all  $g, h \in E$ . But  $E_A(a\lambda(g)) = \tau(a)\lambda(g)$ , for all  $a \in F$  and  $g \in \Gamma$ , and  $E_A(a\lambda(g)\lambda(\gamma)b\lambda(h)) = \tau(a\alpha_\gamma(\alpha_g(b)))\lambda(g\gamma h)$  which suffices to get the conclusion.

Conversely, if  $G$  is a subgroup of  $U(A)$  whose action is weakly mixing relative to  $A$ , let  $F$  be as above and let  $\varepsilon > 0$ . There exists a unitary  $u \in G$  such that

$$\sum_{a,b \in F} \|E_A(aub) - E_A(a)uE_A(b)\|_2^2 < \frac{\varepsilon^2}{2}.$$

As above, since  $u = \sum_\gamma u(\gamma)\lambda(\gamma)$  with  $u(\gamma) \in \mathbb{C}$  for every  $\gamma$ , then

$$E_A(aub) = \sum_\gamma u(\gamma)\tau(a\alpha_\gamma(b))\lambda(\gamma)$$

and

$$E_A(a)uE_A(b) = \sum_\gamma u(\gamma)\tau(a)\tau(b)\lambda(\gamma).$$

This implies that

$$\sum_{a,b \in F} \sum_\gamma |u(\gamma)|^2 |\tau(a\alpha_\gamma(b)) - \tau(a)\tau(b)|^2 < \frac{\varepsilon^2}{2}.$$

As  $\sum_\gamma |u(\gamma)|^2 = 1$ , this implies the existence of  $\gamma \in \Gamma_0$  such that

$$\sum_{a,b \in F} |\tau(a\alpha_\gamma(b)) - \tau(a)\tau(b)|^2 < \varepsilon^2. \quad \square$$

When  $M = L(\Gamma)$  and  $A = L(\Gamma_0)$ , where  $\Gamma_0$  is an abelian subgroup of  $\Gamma$ , it turns out that weak mixing of  $A$  is equivalent to a combinatorial property of the pair of groups  $(\Gamma, \Gamma_0)$  as given by Robertson *et al* [14].

**PROPOSITION 2.3.** *For a pair  $(\Gamma, \Gamma_0)$  as above, the following two conditions are equivalent:*

- (1) (SS) *For every finite subset  $C \subset \Gamma \setminus \Gamma_0$ , there exists  $\gamma \in \Gamma_0$  such that  $g\gamma h \notin \Gamma_0$  for all  $g, h \in C$ ;*
- (2) (WM)  *$A = L(\Gamma_0)$  is weakly mixing in  $M = L(\Gamma)$ .*

*Proof.* If  $(\Gamma, \Gamma_0)$  satisfies condition (SS), take  $G = \lambda(\Gamma_0)$ . Let  $F$  be a finite subset of  $L(\Gamma)$  such that  $E_A(x) = 0$  for every  $x \in F$ . By density, one assumes that  $F$  is contained in  $L_f(\Gamma)$ . We intend to prove that there exists  $\gamma \in \Gamma_0$  such that  $E_A(x\lambda(\gamma)y) = 0$  for all  $x, y \in F$ . By the above assumptions, there exists a finite set  $C \subset \Gamma \setminus \Gamma_0$  such that every  $x \in F$  has support in  $C$ . Fix  $x$  and  $y$  in  $F$  and choose  $\gamma \in \Gamma_0$  as in condition (SS) with respect to  $C$ . Then

$$\begin{aligned} E_A(x\lambda(\gamma)y) &= E_A\left(\sum_{g,h \in C} x(g)y(h)\lambda(g\gamma h)\right) \\ &= \sum_{g,h \in C, g\gamma h \in \Gamma_0} x(g)y(h)\lambda(g\gamma h) = 0 \end{aligned}$$

since the set  $\{(g, h) \in C \times C : g\gamma h \in \Gamma_0\}$  is empty. This shows that  $A$  is weakly mixing in  $M$ .

Conversely, if  $A$  satisfies the latter condition in  $M$ , choose a subgroup  $G$  of  $U(A)$  such that its action on  $M$  is weakly mixing relative to  $A$  and let  $C \subset \Gamma \setminus \Gamma_0$  be a finite set. Take  $F = \lambda(C)$ , and observe that  $E_A(\lambda(g)) = 0$  for every  $g \in C$ . Finally, choose any  $0 < \varepsilon < 1/2$ . Then there exists  $v \in G$  such that

$$\sum_{g, h \in C} \|E_A(\lambda(g)v\lambda(h))\|_2^2 < \varepsilon^2.$$

Now  $v = \sum_{\gamma \in \Gamma_0} v(\gamma)\lambda(\gamma)$  and  $\|v\|_2^2 = 1 = \sum_{\gamma} |v(\gamma)|^2$ . Moreover,

$$\begin{aligned} E_A(\lambda(g)v\lambda(h)) &= \sum_{\gamma \in \Gamma_0} v(\gamma)E_A(\lambda(g\gamma h)) \\ &= \sum_{\gamma \in \Gamma_0, g\gamma h \in \Gamma_0} v(\gamma)\lambda(g\gamma h). \end{aligned}$$

Thus,

$$\|E_A(\lambda(g)v\lambda(h))\|_2^2 = \sum_{\gamma \in \Gamma_0, g\gamma h \in \Gamma_0} |v(\gamma)|^2.$$

Assume that for every  $\gamma \in \Gamma_0$  one can find  $g_\gamma$  and  $h_\gamma$  in  $C$  such that  $g_\gamma\gamma h_\gamma \in \Gamma_0$ . Then we would get

$$\begin{aligned} 1 = \sum_{\gamma \in \Gamma_0} |v(\gamma)|^2 &= \sum_{\gamma \in \Gamma_0, g_\gamma\gamma h_\gamma \in \Gamma_0} |v(\gamma)|^2 \\ &\leq \sum_{g, h \in C} \sum_{\gamma \in \Gamma_0, g\gamma h \in \Gamma_0} |v(\gamma)|^2 < \varepsilon^2, \end{aligned}$$

which is a contradiction.  $\square$

Before concluding the present section, let us recall examples of pairs  $(\Gamma, \Gamma_0)$  that satisfy condition (SS). These are taken from [14]. Let  $\Gamma$  be a group of isometries of some metric space  $(X, d)$  and let  $\Gamma_0$  be an abelian subgroup of  $\Gamma$ . Assume that there is a  $\Gamma_0$ -invariant subset  $Y$  of  $X$  such that:

- (C1) there exists a compact set  $K \subset Y$  such that  $\Gamma_0 K = Y$ ;
- (C2) if  $Y \subset_\delta g_1 Y \cup g_2 Y \cup \dots \cup g_n Y$  for some  $g_j$ s in  $\Gamma$ , and some  $\delta > 0$ , then there exists  $j$  such that  $g_j \in \Gamma_0$ . (Recall that for subsets  $P, Q$  of  $X$  and  $\delta > 0$ ,  $P \subset_\delta Q$  means that  $d(p, Q) \leq \delta$  for every  $p \in P$ .)

Then it is proved in [14, Proposition 4.2] that the pair  $(\Gamma, \Gamma_0)$  satisfies condition (SS).

Now let  $G$  be a semisimple Lie group with no centre and no compact factors. Let  $\Gamma$  be a torsion-free cocompact lattice in  $G$ . Then  $\Gamma$  acts freely on the symmetric space  $X = G/K$ , where  $K$  is a maximal compact subgroup of  $G$ , and the quotient manifold  $M = \Gamma \backslash X$  has universal covering space  $X$ . In particular,  $\Gamma$  is the fundamental group  $\pi(M)$  of  $M$ . Let  $r$  be the rank of  $X$  and let  $T^r \subset M$  be a totally geodesic embedding of a flat  $r$ -torus in  $M$ , so that the inclusion  $i : T^r \rightarrow M$  induces an injective homomorphism  $i_* : \pi(T^r) \rightarrow \pi(M)$ . Thus  $\Gamma_0 = i_*\pi(T^r) \cong \mathbb{Z}^r$  is an abelian subgroup of  $\Gamma$  in a natural way. Then it is proved in [14, Theorem 4.9] that the pair  $(\Gamma, \Gamma_0)$  satisfies conditions (C1) and (C2) above, hence that it satisfies condition (SS) as well. In the same vein, the authors also obtain examples of pairs coming from groups acting cocompactly on locally finite Euclidean buildings.

## 3. Strong mixing

Let  $F$  be Thompson's group, which admits the presentation

$$F = \langle x_0, x_1, \dots \mid x_i^{-1} x_n x_i = x_{n+1}, 0 \leq i < n \rangle.$$

Let  $\Gamma_0$  be the subgroup generated by  $x_0$ . In [6], the first named author proved that the pair  $(F, \Gamma_0)$  satisfies a stronger property than condition (SS) that was called *condition (ST)* and was described as follows.

**Definition 3.1.** Let  $(\Gamma, \Gamma_0)$  be a pair as above. Then we say that it satisfies condition (ST) if, for every finite subset  $C \subset \Gamma \setminus \Gamma_0$ , there exists a finite subset  $E \subset \Gamma_0$  such that  $g g_0 h \notin \Gamma_0$  for all  $g_0 \in \Gamma_0 \setminus E$  and all  $g, h \in C$ .

**Remark.** Observe that, taking finite unions of exceptional sets  $E$  of  $\Gamma_0$  if necessary, condition (ST) is equivalent to the following condition. *For all  $g, h \in \Gamma \setminus \Gamma_0$ , there exists a finite subset  $E$  of  $\Gamma_0$  such that  $g \gamma h \notin \Gamma_0$  for all  $\gamma \in \Gamma_0 \setminus E$ .*

Since the subset  $\Gamma \setminus \Gamma_0$  is stable under the mapping  $g \mapsto g^{-1}$ , when  $\Gamma_0$  is an infinite cyclic group generated by some element  $t$ , condition (ST) is still equivalent to the following condition. *For all  $g, h \in \Gamma \setminus \Gamma_0$ , there exists a positive integer  $N$  such that, for every  $k > N$ ,  $g t^k h \notin \Gamma_0$ .*

For future use in the present section, for every subset  $S$  of a group  $\Gamma$  we put  $S^* = S \setminus \{1\}$ .

We observe in the next section that condition (ST) is strictly stronger than condition (SS); examples are taken from [18, §5]. As is the case for condition (SS), it turns out that condition (ST) is completely characterized by the pair of von Neumann algebras  $L(\Gamma_0) \subset L(\Gamma)$ .

To prove that, we need some facts and definitions. Let  $M$  and  $\tau$  be as above. Let us say that a subset  $S$  of  $M$  is  $\tau$ -orthonormal if  $\tau(xy^*) = \delta_{x,y}$  for all  $x, y \in S$ . We will need a weaker notion which is independent of the chosen trace  $\tau$ .

**PROPOSITION 3.2.** *Let  $M$  be a finite von Neumann algebra, let  $\tau$  be a finite trace on  $M$  as above and let  $S$  be an infinite subset of the unitary group  $U(M)$ . The following conditions are equivalent:*

- (1) *for every  $\varphi \in M_*$  and for every  $\varepsilon > 0$ , there exists a finite subset  $F$  of  $S$  such that  $|\varphi(u)| \leq \varepsilon$  for all  $u \in S \setminus F$ ;*
- (2) *for every  $x \in M$  and for every  $\varepsilon > 0$ , there exists a finite set  $F \subset S$  such that  $|\tau(ux)| \leq \varepsilon$  for all  $u \in S \setminus F$ ;*
- (2') *for any trace  $\tau'$  on  $M$ , for every  $x \in M$  and for every  $\varepsilon > 0$ , there exists a finite set  $F \subset S$  such that  $|\tau'(ux)| \leq \varepsilon$  for all  $u \in S \setminus F$ ;*
- (3) *for every  $\tau$ -orthonormal finite set  $\{x_1, \dots, x_N\} \subset M$  and for every  $\varepsilon > 0$  there exists a finite set  $F \subset S$  such that*

$$\sup\{|\tau(ux^*)|; x \in \text{span}\{x_1, \dots, x_N\}, \|x\|_2 \leq 1\} \leq \varepsilon \quad \text{for all } u \in S \setminus F;$$

- (3') *for every trace  $\tau'$  on  $M$ , for every  $\tau'$ -orthonormal finite set  $\{x_1, \dots, x_N\} \subset M$  and for every  $\varepsilon > 0$  there exists a finite set  $F \subset S$  such that*

$$\sup\{|\tau'(ux^*)|; x \in \text{span}\{x_1, \dots, x_N\}, \|x\|_{2,\tau'} \leq 1\} \leq \varepsilon \quad \text{for all } u \in S \setminus F.$$

In particular, if  $S \subset U(M)$  satisfies the above conditions, and if  $\theta$  is a  $*$ -isomorphism of  $M$  onto some von Neumann algebra  $N$ , then  $\theta(S) \subset U(N)$  satisfies the same conditions.

*Proof.* (1)  $\Rightarrow$  (2')  $\Rightarrow$  (2) and (3')  $\Rightarrow$  (3) are trivial.

(2)  $\Rightarrow$  (3'): If  $\tau'$  is a trace on  $M$ , if  $\{x_1, \dots, x_N\} \subset M$  is  $\tau'$ -orthonormal and if  $\varepsilon > 0$  is fixed, there exists  $h \in M$  such that

$$\|\tau' - \tau(h \cdot)\| \leq \frac{\varepsilon}{2\sqrt{N} \cdot \max \|x_j\|}.$$

Furthermore, one can find a finite set  $F \subset S$  such that

$$|\tau(ux_j^*h)| \leq \frac{\varepsilon}{2\sqrt{N}} \quad \text{for all } u \in S \setminus F \text{ and for all } j = 1, \dots, N.$$

This implies that

$$|\tau'(ux_j^*)| \leq \frac{\varepsilon}{\sqrt{N}} \quad \text{for all } u \in S \setminus F \text{ and for all } j = 1, \dots, N.$$

Let  $x \in \text{span}\{x_1, \dots, x_N\}$ ,  $\|x\|_{2,\tau'} \leq 1$ . Let us write  $x = \sum_{j=1}^N \xi_j x_j$ , where  $\xi_j = \tau'(x x_j^*)$ , and  $\sum_{j=1}^N |\xi_j|^2 = \|x\|_{2,\tau'}^2 \leq 1$  since the  $x_j$ s are  $\tau'$ -orthonormal. Hence we get, for  $u \in S \setminus F$ ,

$$|\tau'(ux^*)| = \left| \sum_{j=1}^N \overline{\xi_j} \tau'(ux_j^*) \right| \leq \left( \sum_{j=1}^N |\xi_j|^2 \right)^{1/2} \left( \sum_{j=1}^N |\tau'(ux_j^*)|^2 \right)^{1/2} \leq \varepsilon$$

uniformly on the set  $\{x \in \text{span}\{x_1, \dots, x_N\}; \|x\|_{2,\tau'} \leq 1\}$ .

(3)  $\Rightarrow$  (1): Let  $\varphi \in M_\star$  and  $\varepsilon > 0$ . We choose  $x \in M$  such that  $\|\varphi - \tau(\cdot x)\| \leq \varepsilon/2$ . Applying condition (3) to the singleton set  $\{x/\|x\|_2\}$  as orthonormal set, we find a finite subset  $F$  of  $S$  such that  $|\tau(ux)| \leq \varepsilon/2$  for every  $u \in S \setminus F$ . Hence we get  $|\varphi(u)| \leq \varepsilon$  for all  $u \in S \setminus F$ .

The last statement follows readily from condition (1).  $\square$

**Definition 3.3.** Let  $M$  be a finite von Neumann algebra endowed with some fixed finite trace  $\tau$ . We say that an infinite subset  $S \subset U(M)$  is *almost orthonormal* if it satisfies the equivalent conditions in Proposition 3.2.

**Remarks.** (1) Since  $M$  has separable predual, an almost orthonormal subset  $S$  of  $U(M)$  is necessarily countable. Indeed, let  $\{x_n; n \geq 1\}$  be a  $\|\cdot\|_2$ -dense countable subset of the unit ball of  $M$  with respect to the operator norm. For  $n \geq 1$ , put

$$S_n = \left\{ u \in S; \max_{1 \leq j \leq n} |\tau(ux_j^*)| \geq \frac{1}{n} \right\}.$$

Then each  $S_n$  is finite,  $S_n \subset S_{n+1}$  for every  $n$  and  $S = \bigcup_n S_n$ . Thus, if  $S = (u_n)_{n \geq 1}$  is a sequence of unitary elements, then  $S$  is almost orthonormal if and only if  $u_n$  tends weakly to zero. In particular, every diffuse von Neumann algebra contains almost orthonormal sequences of unitaries.

(2) The reason why we choose the above definition comes from the fact that if  $S$  is almost orthonormal in  $M$ , then for every  $u \in S$ , for every  $\varepsilon > 0$ , there exists a finite



set  $F \subset S$  such that  $|\tau(v^*u)| < \varepsilon$  for all  $v \in S \setminus F$ . A typical example of an almost orthonormal subset in a finite von Neumann algebra is a  $\tau$ -orthonormal subset  $S$  of  $U(M)$  for *some* trace  $\tau$  on  $M$ : indeed, for every  $x \in M$ , the series  $\sum_{u \in S} |\tau(xu^*)|^2$  converges. For instance, let  $v \in U(M)$  be such that  $\tau(v^k) = 0$  and for all integers  $k \in \mathbb{Z} \setminus \{0\}$ . Then the subgroup generated by  $v$  is almost orthonormal. As another example, let  $\Gamma$  be a countable group and let  $\Gamma_1$  be an infinite subgroup of  $\Gamma$ . Set  $G = \lambda(\Gamma_1)$ . Then  $G$  is almost orthonormal in  $M = L(\Gamma)$ . Indeed, if  $x \in L(\Gamma)$ , then  $\tau(\lambda(g)x) = \tau(x\lambda(g)) = x(g^{-1})$  obviously tends to 0 as  $g$  tends to infinity of  $\Gamma_1$ .

We come now to the main definition of this paper. To motivate it, recall that if  $\Gamma$  is a (countable) group and if  $\alpha$  is a  $\tau$ -preserving action of  $\Gamma$  on  $M$ , then it is called *strongly mixing* if, for every finite set  $F \subset M$  and for every  $\varepsilon > 0$ , there exists a finite set  $E \subset \Gamma$  such that

$$|\tau(\alpha_g(a)b) - \tau(a)\tau(b)| < \varepsilon$$

for all  $a, b \in F$  and all  $g \notin E$ .

**Definition 3.4.** Let  $M$  and  $\tau$  be as above and let  $A$  be an abelian, unital von Neumann subalgebra of  $M$ . Let  $G$  be a subgroup of  $U(A)$ . We say that the action of  $G$  is *strongly mixing relative to  $A$*  if, for all  $x, y \in M$ ,

$$\lim_{u \rightarrow \infty, u \in G} \|E_A(uxu^{-1}y) - E_A(x)E_A(y)\|_2 = 0.$$

Furthermore, we say that  $A$  itself is *strongly mixing in  $M$*  if, for every almost orthonormal infinite subgroup  $G$  of  $U(A)$ , the action of  $G$  by inner automorphisms on  $M$  is strongly mixing relative to  $A$ .

**Remark.** The above property is independent of the trace  $\tau$  and it is a conjugacy invariant. Indeed,  $E_A$  is the unique conditional expectation onto  $A$  and almost orthonormality is independent of the chosen trace.

We now present our main result.

**THEOREM 3.5.** *Let  $\Gamma$  be an infinite group and let  $\Gamma_0$  be an infinite abelian subgroup of  $\Gamma$ . Let  $M = L(\Gamma)$  and  $A = L(\Gamma_0)$  be as above. Then the following conditions are equivalent:*

- (1) *the action of  $\Gamma_0$  by inner automorphisms on  $M$  is strongly mixing relative to  $A$ ;*
- (2) *for every finite subset  $C \subset \Gamma \setminus \Gamma_0$  there exists a finite subset  $E \subset \Gamma_0$  such that  $gg_0h \notin \Gamma_0$  for all  $g_0 \in \Gamma_0 \setminus E$  and all  $g, h \in C$ —that is, the pair  $(\Gamma, \Gamma_0)$  satisfies condition (ST);*
- (3) *for every almost orthonormal infinite subset  $S \subset U(A)$ , for all  $x, y \in M$  and for every  $\varepsilon > 0$ , there exists a finite subset  $F \subset S$  such that*

$$\|E_A(uxu^*y) - E_A(x)E_A(y)\|_2 < \varepsilon \quad \text{for all } u \in S \setminus F;$$

- (4)  *$A$  is strongly mixing in  $M$ .*

*Proof.* Trivially, (3)  $\Rightarrow$  (4)  $\Rightarrow$  (1).

(1)  $\Rightarrow$  (2): If  $C$  is as in (2), set

$$x = \sum_{g \in C} \lambda(g).$$

Thus  $E_A(x) = 0$ , and there exists a finite subset  $E \subset \Gamma_0$  such that

$$\|E_A(\lambda(g_0^{-1})x\lambda(g_0)x)\|_2 = \|E_A(x\lambda(g_0)x)\|_2 < 1 \quad \text{for all } g_0 \in \Gamma_0 \setminus E.$$

But

$$E_A(x\lambda(g_0)x) = E_A\left(\sum_{g \in C, h \in D} \lambda(gg_0h)\right) = \sum_{g, h \in C, gg_0h \in \Gamma_0} \lambda(gg_0h),$$

hence  $\|E_A(x\lambda(g_0)x)\|_2^2 = |\{(g, h) \in C \times C; gg_0h \in \Gamma_0\}| < 1$  for all  $g_0 \notin E$ , which implies that  $gg_0h \notin \Gamma_0$  for  $g_0 \notin E$  and for all  $g, h \in C$ .

(2)  $\Rightarrow$  (3): Let  $S$  be an almost orthonormal infinite subset of  $U(A)$ , let  $x, y \in M$  and fix  $\varepsilon > 0$ . Using the  $A$ -bilinearity of  $E_A$ , we can assume that  $E_A(x) = E_A(y) = 0$ , and we have to prove that we can find a finite set  $F \subset S$  such that  $\|E_A(uxu^*y)\|_2 < \varepsilon$  for all  $u \in S \setminus F$ . To begin with, let us assume furthermore that  $x$  and  $y$  have finite support, and let us write  $x = \sum_{g \in C} x(g)\lambda(g)$  and  $y = \sum_{h \in C} y(h)\lambda(h)$  with  $C \subset \Gamma \setminus \Gamma_0$  finite. Let  $E \subset \Gamma_0$  be as in (2) with respect to the finite set  $C$  of  $\Gamma \setminus \Gamma_0$ , namely  $gg_0h \notin \Gamma_0$  for all  $g, h \in C$ , and  $g_0 \notin E$ . We claim then that  $E_A(x\lambda(g_0^{-1})y) = 0$  if  $g_0 \in \Gamma_0 \setminus E^{-1}$ . Indeed, if  $g_0 \in \Gamma_0 \setminus E^{-1}$ , then

$$\begin{aligned} E_A(x\lambda(g_0^{-1})y) &= E_A\left(\sum_{g, h \in C} x(g)y(h)\lambda(gg_0^{-1}h)\right) \\ &= \sum_{g, h \in C, gg_0^{-1}h \in \Gamma_0} x(g)y(h)\lambda(gg_0^{-1}h) = 0 \end{aligned}$$

because  $g_0^{-1} \in \Gamma_0 \setminus E$ .

Choosing  $\lambda(E) \subset A$  as a  $\tau$ -orthonormal system, there exists a finite subset  $F$  of  $S$  such that, if  $u \in S \setminus F$ ,

$$\sup\{|\tau(uz^*)|; z \in \text{span}(\lambda(E)), \|z\|_2 \leq 1\} < \frac{\varepsilon^2}{|E|\|x\|\|y\|}.$$

Thus, for fixed  $u \in S \setminus F$ , take  $z := \sum_{g_0 \in E} u(g_0)\lambda(g_0)$ , so that  $z \in \text{span}(\lambda(E))$ ,  $\|z\|_2 \leq 1$  and

$$\sum_{g_0 \in E} |u(g_0)|^2 = \tau(uz^*) < \frac{\varepsilon^2}{|E|\|x\|\|y\|}.$$

Then

$$\begin{aligned} \|E_A(uxu^*y)\|_2 &= \|uE_A(xu^*y)\|_2 = \|E_A(xu^*y)\|_2 \\ &\leq \sum_{g_0 \in E} |u(g_0)| \|E_A(x\lambda(g_0^{-1})y)\|_2 < \varepsilon, \end{aligned}$$

using the Cauchy–Schwarz inequality.

Finally, if  $x, y \in M$  are such that  $E_A(x) = E_A(y) = 0$ , if  $\varepsilon > 0$ , let  $x', y' \in L_f(\Gamma)$  be such that  $\|x'\| \leq \|x\|$ ,  $\|y'\| \leq \|y\|$ ,  $E_A(x') = E_A(y') = 0$  and

$$\|x' - x\|_2, \|y' - y\|_2 < \frac{\varepsilon}{3 \cdot \max(\|x\|, \|y\|)}.$$

Take a finite subset  $F \subset S$  such that  $\|E_A(ux'u^*y')\|_2 < \varepsilon/3$  for every  $u \in S \setminus F$ . Then, if  $u \in S \setminus F$ ,

$$\|E_A(uxu^*y)\|_2 < \|y\|\|x - x'\|_2 + \|x\|\|y - y'\|_2 + \frac{\varepsilon}{3} < \varepsilon.$$

This concludes the proof of Theorem 3.5.  $\square$

As in the case of weak mixing, the following proposition holds for crossed products.

**PROPOSITION 3.6.** *Let  $\Gamma_0$  be an abelian group which acts on a finite von Neumann algebra  $B$  and which preserves the trace  $\tau$ , then the abelian von Neumann subalgebra  $A = L(\Gamma_0)$  of the crossed product  $M = B \rtimes \Gamma_0$  is strongly mixing in  $M$  if and only if the action of  $\Gamma_0$  is.*

In fact, Proposition 3.6 above and Schmidt [15, Theorem 4.2] prove that weakly mixing MASAs are not strongly mixing in general.

Let  $\Gamma_0$  be an infinite abelian group and let  $\alpha$  be a measure-preserving, free, weakly mixing but not strongly mixing action on some standard probability space  $(X, \mathcal{B}, \mu)$  as in [15, Theorem 4.2]. Set  $B = L^\infty(X, \mathcal{B}, \mu)$  and let  $M$  be the corresponding crossed product  $\Pi_1$ -factor. Then the abelian subalgebra  $A = L(\Gamma_0)$  is a weakly mixing MASA in  $M$ , but it is not strongly mixing.

Typical examples of strongly mixing actions are given by (generalized) Bernoulli shift actions. Consider a finite von Neumann algebra  $B \neq \mathbb{C}$  endowed with some trace  $\tau_B$ . Let  $\Gamma_0$  be an infinite abelian group that acts *properly* on a countable set  $X$ : for every finite set  $Y \subset X$ , the set  $\{g \in \Gamma_0; g(Y) \cap Y \neq \emptyset\}$  is finite. Let  $(N, \tau) = \bigotimes_{x \in X} (B, \tau_B)$  be the associated infinite tensor product. Then the corresponding Bernoulli shift action is the action  $\sigma$  of  $\Gamma_0$  on  $N$  given by

$$\sigma_g \left( \bigotimes_{x \in X} b_x \right) = \bigotimes_{x \in X} b_{gx}$$

for every  $\bigotimes_{x \in X} b_x \in N$  such that  $b_x = 1$  for all but finitely many  $x$ s. Then it is easy to see that properness of the action implies that  $\sigma$  is a strongly mixing action. The classical case corresponds to the simply transitive action by left translations on  $\Gamma_0$ .

Let  $A$  be a diffuse von Neumann subalgebra of  $M$  and let  $Q$  be a finite von Neumann algebra. It is proved in [4, Theorem 2.3] that the normalizers  $\mathcal{N}_M(A)$  and  $\mathcal{N}_{M \star Q}(A)$  are equal. In particular, if  $A$  is a singular MASA in  $M$  then it is also a singular MASA in the free product  $M \star Q$ . Then, using the same arguments as in the proofs of Lemma 2.2, Theorem 2.3 and Corollary 2.4 of [4], we obtain the following proposition.

**PROPOSITION 3.7.** *Let  $M$  and  $Q$  be finite von Neumann algebras and let  $A$  be a strongly mixing MASA in  $M$ . Then  $A$  is also strongly mixing in the free product von Neumann algebra  $M \star Q$ .*

#### 4. Examples

From now on, we consider pairs  $(\Gamma, \Gamma_0)$  where  $\Gamma_0$  is an abelian subgroup of  $\Gamma$ . We will give examples of families of pairs that satisfy condition (ST) on the one hand, and of pairs that satisfy condition (SS) but not (ST) on the other hand.

4.1. *Some free products examples.* It was noted in [6] that if  $\Gamma$  is the free group  $F_N$  of rank  $N \geq 2$  on free generators  $a_1, \dots, a_N$  and if  $\Gamma_0$  is the subgroup generated by some fixed  $a_i$ , then the pair  $(F_N, \Gamma_0)$  satisfies condition (ST); see also [17, Corollary 3.4]. The next result extends the above case to some amalgamated products.

**PROPOSITION 4.1.** *Let  $\Gamma = \Gamma_0 \star_Z \Gamma_1$  be an amalgamated product where  $\Gamma_0$  is an infinite abelian group, and  $Z$  is a finite subgroup of  $\Gamma_0$  and of  $\Gamma_1$ . If  $\Gamma_1 \neq Z$  then the pair  $(\Gamma, \Gamma_0)$  satisfies condition (ST).*

*Proof.* Let  $R$  and  $S$  be sets of representatives for the left cosets of  $Z$  in  $\Gamma_0$  and  $\Gamma_1$  respectively, such that  $1 \in R$  and  $1 \in S$ . Recall that every element  $g \in \Gamma$  has a unique normal form

$$g = r_1 s_1 \dots r_l s_l z$$

with  $r_i \in R$ ,  $s_i \in S$  for every  $i$ , such that only  $r_1$  or  $s_1$  can be equal to 1, and  $z \in Z$ . Notice that  $g$  does not belong to  $\Gamma_0$  if and only if  $s_1 \neq 1$ . Let  $g, h \in \Gamma \setminus \Gamma_0$ . It suffices to find a finite subset  $E \subset \Gamma_0$  such that, for every  $\gamma \in \Gamma_0 \setminus E$ ,  $g\gamma h \notin \Gamma_0$ . Let us write  $g = r_1 s_1 \dots r_l s_l z$  with  $r_i \in R$ ,  $s_i \in S$  and  $z \in Z$ , and  $h = u_1 v_1 \dots u_k v_k w$  with  $u_j \in R$ ,  $v_j \in S$  and  $w \in Z$ . If  $\gamma = rt$  with  $r \in R$  and  $t \in Z$ , then

$$\begin{aligned} g\gamma h &= r_1 s_1 \dots r_l s_l z r t u_1 v_1 \dots u_k v_k w \\ &= r_1 s_1 \dots r_l s_l (r u_1) v'_1 z'_1 \dots u_k v_k w \end{aligned}$$

with  $v'_1 \in S$  and  $z'_1 \in Z$  such that  $v'_1 z'_1 = (zt)v_1$  is the decomposition of  $(zt)v_1$  in the partition  $\Gamma_1 = \coprod_{\sigma \in S} \sigma Z$ . Observe that  $v'_1 \neq 1$  because  $v_1 \neq 1$ . Continuing in the same way, we move elements of  $Z$  to the right as far as possible and we finally obtain

$$g\gamma h = r_1 s_1 \dots r_l s_l (r u_1) v'_1 \dots u_k v'_k w'$$

with every  $v'_j \in S$  and  $w' \in Z$ . If  $s_l \neq 1$  we can take  $E = Z \cup u_1^{-1}Z$ , and if  $s_l = 1$  we can take  $E = Z \cup (r_l u_1)^{-1}Z$ .  $\square$

Finally, from Theorem 3.5 and Proposition 3.7 we obtain the following proposition.

**PROPOSITION 4.2.** *Let  $\Gamma$  be an infinite group, let  $\Gamma_0$  be an infinite abelian subgroup of  $\Gamma$  such that the pair  $(\Gamma, \Gamma_0)$  satisfies condition (ST) and let  $G$  be any countable group. Then the pair  $(\Gamma \star G, \Gamma_0)$  also satisfies condition (ST).*

4.2. *The case of malnormal subgroups.* Popa [10] introduced a condition on pairs  $(\Gamma, \Gamma_0)$  in order to obtain orthogonal pairs of von Neumann subalgebras; it was also used later in [17] to get asymptotic homomorphism conditional expectations. We say that  $\Gamma_0$  is a *malnormal* subgroup of  $\Gamma$  if it satisfies the following condition:

( $\star$ ) *For every  $g \in \Gamma \setminus \Gamma_0$ ,  $g\Gamma_0 g^{-1} \cap \Gamma_0 = \{1\}$ .* Then we observe that condition ( $\star$ ) implies condition (ST). Indeed, for  $g, h \in \Gamma$ , set  $E(g, h) = \{\gamma \in \Gamma_0; g\gamma h \in \Gamma_0\} = g^{-1}\Gamma_0 h^{-1} \cap \Gamma_0$ . If  $(\Gamma, \Gamma_0)$  satisfies ( $\star$ ), and if  $g, h \in \Gamma \setminus \Gamma_0$ , then  $E(g, h)$  contains at most one element (see the proof of [17, Lemma 3.1]). In turn, condition (ST) means exactly that  $E(g, h)$  is finite for all  $g, h \in \Gamma \setminus \Gamma_0$ . Thus, if  $\Gamma_0$  is torsion-free, then

conditions  $(\star)$  and (ST) are equivalent because, in this case, if  $g \in \Gamma \setminus \Gamma_0$ , the finite set  $E(g^{-1}, g) = g\Gamma_0g^{-1} \cap \Gamma_0$  is a finite subgroup of  $\Gamma_0$ , hence is trivial.

However, condition  $(\star)$  is strictly stronger than condition (ST) in general. Let  $\Gamma = \Gamma_0 \star_Z \Gamma_1$  be as in Proposition 4.1 above, and assume further that  $Z \neq \{1\}$  and that there exists  $g \in \Gamma_1 \setminus Z$  such that  $zg = gz$  for every  $z \in Z$ . Then  $g\Gamma_0g^{-1} \cap \Gamma_0 \supset Z$ , and  $(\Gamma, \Gamma_0)$  does not satisfy  $(\star)$ . Observe that, in this case,  $\Gamma$  is not necessarily an Infinite Conjugacy Classes (ICC) group. However, replacing it by a non-trivial free product group  $\Gamma \star G$ , we get a pair  $(\Gamma \star G, \Gamma_0)$  satisfying condition (ST) by Proposition 3.7 but not  $(\star)$ , and  $\Gamma \star G$  is an ICC group.

It is known that in some classes of groups, maximal abelian subgroups are malnormal. This is the case, for example, in hyperbolic groups [5] or in groups acting freely on  $\Lambda$ -trees [1]. We present here explicitly a sufficient condition to get malnormal subgroups in groups acting on trees.

**PROPOSITION 4.3.** *Let  $\Gamma$  be a group acting on a tree  $T$  without inversion. Let  $t \in \Gamma$  and set  $\Gamma_0 = \langle t \rangle$ . Assume that:*

- (1) *there exist neither  $u \in \Gamma$  nor  $n \in \mathbb{Z} \setminus \{\pm 1\}$  such that  $t = u^n$ ;*
- (2) *the induced automorphism of  $T$  (again denoted by  $t$ ) is hyperbolic;*
- (3) *the subgroup  $\bigcap_{v \in \text{axis}(t)} \text{Stab}(v)$  is trivial;*
- (4) *no element of  $\Gamma$  induces a reflection of  $\text{axis}(t)$ .*

*Then  $\Gamma_0$  is a malnormal subgroup of  $\Gamma$ , and in particular the pair  $(\Gamma, \Gamma_0)$  satisfies condition (ST).*

Before proving this, we introduce the necessary terminology for actions on trees. If a group  $G$  acts without inversion on a tree  $X$ , it is well known [16, Ch. I.6.4] that an element  $g \in G$  is either *elliptic* (that is, it fixes a vertex) or *hyperbolic* (that is,  $g$  preserves an infinite geodesic, called an *axis*, on which it acts by a non-trivial translation). It is easy to see that a hyperbolic element  $g$  has a unique axis. It will be denoted  $\text{axis}(g)$ .

*Proof of Proposition 4.3.* Given an element  $g \in \Gamma$  such that  $g\Gamma_0g^{-1} \cap \Gamma_0 \neq \{1\}$ , we have to prove that  $g \in \Gamma_0$ . First, we write  $gt^k g^{-1} = t^{k'}$  for some  $k, k' \in \mathbb{Z}^*$ . Since the elements  $t^k$  and  $t^{k'}$  have  $\text{axis}(t)$  as axis,  $g$  has to preserve  $\text{axis}(t)$ . Then, by hypotheses (3) and (4),  $g$  is a hyperbolic element and  $\text{axis}(g) = \text{axis}(t)$ . Let us denote by  $\ell(\gamma)$  the translation length of an element  $\gamma \in \Gamma$ . By Bézout's theorem, there exists an element  $s = g^m t^n$  with  $m, n \in \mathbb{Z}$  such that  $\ell(s)$  is the greatest common divisor of  $\ell(g)$  and  $\ell(t)$ . Then there exist  $\delta, \varepsilon \in \{\pm 1\}$  such that  $gs^{\delta\ell(g)/\ell(s)}$  and  $ts^{\varepsilon\ell(t)/\ell(s)}$  fix every vertex of  $\text{axis}(t)$ . By hypothesis (3), we get  $g = s^{-\delta\ell(g)/\ell(s)}$  and  $t = s^{-\varepsilon\ell(t)/\ell(s)}$ . Then (1) gives  $t = s^{\pm 1}$ , so that  $g = t^{\pm\ell(g)/\ell(t)}$ . As desired,  $g$  is an element of  $\Gamma_0$ .  $\square$

As will be seen below, wide families of HNN extensions satisfy hypotheses of Proposition 4.3. Thus, let  $\Gamma = \text{HNN}(\Lambda, H, K, \phi)$  be an HNN extension where  $H, K$  are subgroups of  $\Lambda$  and, as usual, where  $\phi: H \rightarrow K$  is an isomorphism. Denote by  $t$  the stable letter such that  $t^{-1}ht = \phi(h)$  for all  $h \in H$ , and by  $\Gamma_0$  the subgroup generated by  $t$ . Recall that a sequence  $g_0, t^{\varepsilon_1}, \dots, t^{\varepsilon_n}, g_n$ , ( $n \geq 0$ ) is *reduced* if  $g_i \in \Lambda$  and  $\varepsilon_i = \pm 1$  for every  $i$ , and if there is no subsequence  $t^{-1}, g_i, t$  with  $g_i \in H$  or  $t, g_i, t^{-1}$  with  $g_i \in K$ . As is well

known, if the sequence  $g_0, t^{\varepsilon_1}, \dots, t^{\varepsilon_n}, g_n$  is reduced and if  $n \geq 1$  then the corresponding element  $g = g_0 t^{\varepsilon_1} \dots t^{\varepsilon_n} g_n \in \Gamma$  is non-trivial (Britton's lemma). We also say that such an element is in *reduced form*. Furthermore, if  $g = g_0 t^{\varepsilon_1} \dots t^{\varepsilon_n} g_n$  and  $h = h_0 t^{\delta_1} \dots t^{\delta_m} g_m$  are in reduced form and if  $g = h$ , then  $n = m$  and  $\varepsilon_i = \delta_i$  for every  $i$ . Hence the *length*  $\ell(g)$  of  $g = g_0 t^{\varepsilon_1} \dots t^{\varepsilon_n} g_n$  (in reduced form) is the integer  $n$ . Finally, recall from [20] that, for every positive integer  $j$ ,  $\text{Dom}(\phi^j)$  is defined by  $\text{Dom}(\phi) = H$  for  $j = 1$  and, by induction,  $\text{Dom}(\phi^j) = \phi^{-1}(\text{Dom}(\phi^{j-1}) \cap K) \subset H$  for  $j \geq 2$ .

If  $\Gamma = \text{HNN}(\Lambda, H, K, \phi)$ , its *Bass–Serre tree* has  $\Gamma/\Lambda$  as set of vertices and  $\Gamma/H$  as set of oriented edges. The origin of the edge  $\gamma H$  is  $\gamma \Lambda$  and its terminal vertex is  $\gamma t \Lambda$ . Serre [16, Ch. I.5], and Theorem 12 in particular, ensures that it is a tree. It is obvious that the Bass–Serre tree is endowed with an orientation-preserving  $\Gamma$ -action.

**COROLLARY 4.4.** *Suppose that for each  $\lambda \in \Lambda^*$ , there exists  $j > 0$  such that  $\lambda \notin \text{Dom}(\phi^j)$ . Then  $\Gamma_0$  is a malnormal subgroup of  $\Gamma$ .*

*Proof.* Let  $T$  be the Bass–Serre tree of the HNN extension. We check the hypotheses of Proposition 4.3. Since (1), (2) and (4) are obvious, we prove (3). Now  $\bigcap_{v \in \text{axis}(t)} \text{Stab}(v) = \bigcap_{k \in \mathbb{Z}} \Lambda_k$ , where  $\Lambda_k = t^{-k} \Lambda t^k$ . Assume by way of contradiction that the intersection contains a non-trivial element  $\lambda$ . By hypothesis, there exists  $j \in \mathbb{N}^*$  such that  $\lambda \notin \text{Dom}(\phi^j)$ , and we may assume  $j$  to be minimal for this property. This means that  $t^{1-j} \lambda t^{j-1} = \phi^{j-1}(\lambda) \in \Lambda \setminus H$ , and  $t^{-j} \lambda t^j \notin \Lambda$  by Britton's lemma. We get  $\lambda \notin \Lambda_{-j}$ , a contradiction. This proves (3).  $\square$

As will be recalled in the first example below, Thompson's group  $F$  is an HNN extension, and it satisfies condition (ST) with respect to the subgroup generated by  $x_0$ , by [6, Lemma 3.2].

*Examples.* (1) For every integer  $k \geq 1$ , denote by  $F_k$  the subgroup of  $F$  generated by  $x_k, x_{k+1}, \dots$ , and denote by  $\sigma$  the 'shift map' defined by  $\sigma(x_n) = x_{n+1}$ , for  $n \geq 0$ . Its restriction to  $F_k$  is an isomorphism onto  $F_{k+1}$ , and in particular, the inverse map  $\phi: F_2 \rightarrow F_1$  is an isomorphism which satisfies  $\phi(x) = x_0 x x_0^{-1}$  for every  $x \in F_2$ . As in [3, Proposition 1.7], it is evident that  $F$  is the HNN extension  $\text{HNN}(F_1, F_2, F_1, \phi)$  with  $t = x_0^{-1}$  as stable letter. With these choices,  $F$  satisfies the hypotheses of Proposition 4.1, and this proves that the pair  $(F, \Gamma_0)$  satisfies condition (ST).

(2) Let  $m$  and  $n$  be non-zero integers. The associated *Baumslag–Solitar group* is the group which has the presentation

$$BS(m, n) = \langle a, b \mid ab^m a^{-1} b^{-n} \rangle.$$

Since  $a^{-1} b^n a = b^m$ ,  $BS(m, n)$  is an HNN extension  $\text{HNN}(\mathbb{Z}, n\mathbb{Z}, m\mathbb{Z}, \phi)$  where  $\phi(nk) = mk$  for every integer  $k$ . Assume first that  $|n| > |m|$  and denote by  $\Gamma_0$  the subgroup generated by  $a$ . Then it is easy to check that the pair  $(\Gamma_0, BS(m, n))$  satisfies the condition in Proposition 4.1. Thus, it also satisfies condition (ST). If  $|m| > |n|$ , replacing  $a$  by  $a^{-1}$  (which does not change  $\Gamma_0$ ), one reaches the same conclusion. Thus, all Baumslag–Solitar groups  $BS(m, n)$  with  $|m| \neq |n|$  satisfy condition (ST) with respect to the subgroup generated by  $a$ . Observe that the latter class is precisely the class of Baumslag–Solitar groups that are ICC [20].

We turn to free products. The Bass–Serre tree of  $\Gamma = A * B$  has  $(\Gamma/A) \sqcup (\Gamma/B)$  as set of vertices and  $\Gamma$  as set of oriented edges. The origin of the edge  $\gamma$  is  $\gamma A$  and its terminal vertex is  $\gamma B$ . Again, [16, Ch. I.5] ensures that it is a tree. Here is another consequence of Proposition 4.3.

**COROLLARY 4.5.** *Let  $\Gamma = A * B$  be a free product such that  $|A| \geq 2$  and  $B$  contains an element  $b$  of order at least 3. Let  $a$  be a non-trivial element of  $A$  and  $\Gamma_0 = \langle ab \rangle$ . Then  $\Gamma_0$  is malnormal in  $\Gamma$ .*

*Proof.* Let  $T$  be the Bass–Serre tree of the free product. Again, we check the hypotheses of Proposition 4.3, with  $t = ab$ . Since (1) and (2) are obvious, we focus on (3) and (4). The axis of  $t$  has the structure

$$\dots \leftarrow b^{-1}a^{-1}b^{-1}A \rightarrow b^{-1}a^{-1}B \leftarrow b^{-1}A \rightarrow B \leftarrow A \rightarrow aB \leftarrow abA \rightarrow \dots$$

To prove (3), we remark that the vertices  $A$  and  $abA$  are on  $\text{axis}(t)$  and that  $\text{Stab}(A) \cap \text{Stab}(abA) = A \cap abAb^{-1}a^{-1} = \{1\}$ .

To prove (4), assume by contradiction that there exists an elliptic element  $g \in \Gamma$  which induces a reflection on  $\text{axis}(t)$ . Up to conjugating  $g$  by a power of  $t$ , we may assume that the fixed point of  $g$  on  $\text{axis}(t)$  is  $A$  or  $B$ . If  $g$  fixes  $A$ , then  $gb^{-1}A = abA$ , so that  $g \in A$  and  $bg^{-1}ab \in A$ . This is a contradiction since  $|b| \geq 3$ . Now, if  $g$  fixes  $B$ , then  $gabA = b^{-1}a^{-1}b^{-1}A$ , so that  $g \in B$  and  $babgab \in A$ . This implies  $g = b^{-1}$ ,  $a^2 = 1$ , and  $b^2 = 1$ , which is again a contradiction since  $|b| \geq 3$ .  $\square$

**4.3. Some semidirect products.** Let  $H$  be a discrete group, let  $\Gamma_0$  be an infinite abelian group and let  $\alpha : \Gamma_0 \rightarrow \text{Aut}(H)$  be an action of  $\Gamma_0$  on  $H$ . Then the semi-direct product  $\Gamma = H \rtimes_{\alpha} \Gamma_0$  is the direct product set  $H \times \Gamma_0$  endowed with multiplication

$$(h, \gamma)(h', \gamma') = (h\alpha_{\gamma}(h'), \gamma\gamma') \quad \text{for all } (h, \gamma), (h', \gamma') \in \Gamma.$$

The action  $\alpha$  lifts from  $H$  to the von Neumann algebra  $L(H)$ , and  $L(\Gamma) = L(H) \rtimes \Gamma_0$  is a crossed product von Neumann algebra in a natural way. Robertson *et al* [14, Theorem 2.2] consider a sufficient condition on the action  $\alpha$  on  $H$  which ensures that  $L(\Gamma_0) \subset L(\Gamma)$  is a strongly singular MASA in  $L(\Gamma)$  and that  $L(\Gamma)$  is a type  $\text{II}_1$  factor. In fact, it turns out that their condition implies that  $L(\Gamma_0)$  is strongly mixing in  $L(\Gamma)$ , as we now prove.

**PROPOSITION 4.6.** *Let  $H$  and  $\Gamma_0$  be infinite discrete groups,  $\Gamma_0$  being abelian, let  $\alpha$  be an action of  $\Gamma_0$  on  $H$  and let  $\Gamma = H \rtimes_{\alpha} \Gamma_0$ . If, for each  $\gamma \in \Gamma_0^*$ , the only fixed point of  $\alpha_{\gamma}$  is  $1_H$ , then:*

- (1)  $\Gamma$  is an ICC group;
- (2) the pair  $(\Gamma, \Gamma_0)$  satisfies condition (ST);
- (3) the action of  $\Gamma_0$  on  $L(H)$  is strongly mixing.

*In particular,  $L(\Gamma)$  is a type  $\text{II}_1$  factor and  $L(\Gamma_0)$  is strongly mixing in  $L(\Gamma)$ .*

*Proof.* Statement (1) is proved in [14]. Thus it remains to prove (2) and (3).

To do that, we claim first that if  $\alpha$  is as above, then the triple  $(\Gamma_0, H, \alpha)$  satisfies the following condition whose proof is inspired by that of [14, Theorem 2.2]. For every



finite subset  $F$  of  $H^*$ , there exists a finite set  $E$  in  $\Gamma_0$  such that  $\alpha_\gamma(F) \cap F = \emptyset$  for all  $\gamma \in \Gamma_0 \setminus E$ .

Indeed, if  $F$  is fixed, set  $I(F) = \{\gamma \in \Gamma_0; \alpha_\gamma(F) \cap F \neq \emptyset\}$ . If  $I(F)$  is infinite for some  $F$ , then set, for  $f \in F$ ,

$$S_f = \{\gamma \in \Gamma_0; \alpha_\gamma(f) \in F\},$$

so that  $I(F) = \bigcup_{f \in F} S_f$ , and  $S_f$  would be infinite for at least one  $f \in F$ . There would then exist distinct elements  $\gamma_1$  and  $\gamma_2$  of  $\Gamma_0$  such that  $\alpha_{\gamma_1}(f) = \alpha_{\gamma_2}(f)$ , since  $F$  is finite. This is impossible because  $\alpha_{\gamma_1^{-1}\gamma_2}$  cannot have any fixed point in  $F \subset H^*$ .

Let us prove (2). Fix some finite set  $C \subset \Gamma \setminus \Gamma_0$ . Without loss of generality, we assume that  $C = C_1 \times C_2$  with  $C_1 \subset H^*$  and  $C_2 \subset \Gamma_0$  finite. Take  $F = C_1 \cup C_1^{-1} \subset H^*$  in the condition above and let  $E_1 \subset \Gamma_0$  be a finite set such that  $\alpha_\gamma(F) \cap F = \emptyset$  for all  $\gamma \in \Gamma_0 \setminus E_1$ . Put  $E = \bigcup_{\gamma \in C_2} \gamma^{-1}E_1$ , which is finite. Then it is easy to check that for all  $(h, \gamma), (h', \gamma') \in C$  and for every  $g \in \Gamma_0 \setminus E$ ,

$$(h, \gamma)(1, g)(h', \gamma') = (h\alpha_{\gamma g}(h'), \gamma g\gamma') \notin \{1_H\} \times \Gamma_0.$$

Finally, in order to prove (3), it suffices to see that, if  $a, b \in L_f(H)$ , then there exists a finite subset  $E$  of  $\Gamma_0$  such that

$$\tau(\alpha_\gamma(a)b) = \tau(a)\tau(b) \quad \text{for all } \gamma \notin E.$$

Thus, let  $S \subset \Gamma_0$  be a finite subset such that  $a = \sum_{\gamma \in S} a(\gamma)\lambda(\gamma)$  and  $b = \sum_{\gamma \in S} b(\gamma)\lambda(\gamma)$ . Choose  $E \subset \Gamma_0$  finite such that  $\alpha_\gamma(S^* \cup (S^*)^{-1}) \cap (S^* \cup (S^*)^{-1}) = \emptyset$  for every  $\gamma \in \Gamma_0 \setminus E$ . Then, if  $\gamma \notin E$ ,

$$\tau(\alpha_\gamma(a)b) = \sum_{h_1, h_2 \in S} a(h_1)b(h_2)\tau(\lambda(\alpha_\gamma(h_1)h_2)) = \tau(a)\tau(b)$$

since  $\alpha_\gamma(h_1)h_2 \neq 1$  for  $h_1, h_2 \in S^*$ . □

*Example.* Let  $d \geq 2$  be an integer and let  $g \in GL(d, \mathbb{Z})$ . This defines a natural action of  $\Gamma_0 = \mathbb{Z}$  on  $H = \mathbb{Z}^d$  which has no non-trivial fixed point if and only if the list of eigenvalues of  $g$  contains no root of unity (see, for instance, [20, Example 2.5]).

We give below an application of Proposition 4.6 to some HNN extensions (a wider class than in Corollary 4.4); as follows from [16, Ch. I.1.4, Proposition 5], if  $\Gamma = HNN(\Lambda, H, K, \phi)$  with stable letter  $t$  such that  $t^{-1}ht = \phi(h)$  for all  $h \in H$ , if  $\Gamma_0 = \langle t \rangle$  as above and if  $\sigma_t$  denotes the quotient map  $\Gamma \rightarrow \Gamma_0$ , then  $\Gamma$  is a semidirect product group  $N \rtimes \Gamma_0$  where  $N = \ker(\sigma_t)$  is a direct limit, or *amalgam*, of the system

$$\dots \Lambda_{-1} \searrow K_{-1} = H_0 \nearrow \Lambda_0 \searrow K_0 = H_1 \nearrow \Lambda_1 \dots$$

where  $\Lambda_i = t^{-i}\Lambda t^i$ ,  $H_i = t^{-i}Ht^i$  and  $K_i = t^{-i}Kt^i$  for all  $i \in \mathbb{Z}$ . The conjugation operation  $n \mapsto t^{-1}nt$  then corresponds to a shift to the right.

**COROLLARY 4.7.** *Suppose that for all  $j \in \mathbb{N}^*$ , the homomorphism  $\phi^j$  has no non-trivial fixed point, that is, for all  $h \in H$ ,  $\phi^j(h) = h$  implies  $h = 1$ . Then the following hold:*



- (1) the group  $\Gamma$  is ICC;
- (2) the pair  $(\Gamma, \Gamma_0)$  satisfies condition (ST);
- (3) the algebra  $L(\Gamma)$  is a type II<sub>1</sub> factor in which  $L(\Gamma_0)$  is strongly mixing.

*Proof.* It suffices to prove that, for any  $k \in \mathbb{Z}^*$  and any  $n \in N^*$ ,  $t^{-k}nt^k \neq n$ .

Assume by way of contradiction that  $t^{-k}nt^k = n$  for some  $k \in \mathbb{Z}^*$ , and some  $n \in N^*$ . Up to replacing  $k$  by  $-k$ , we assume that  $k \in \mathbb{N}^*$ . Then there exists  $s \in \mathbb{N}$  such that  $n$  is in the subgroup of  $N$  generated by  $\Lambda_{-s}, \dots, \Lambda_0, \dots, \Lambda_s$ . Now let  $\ell$  be a multiple of  $k$  such that  $\ell - s > s + k$ . The element  $t^{-\ell}nt^\ell$  is in the subgroup of  $N$  generated by  $\Lambda_{\ell-s}, \dots, \Lambda_\ell, \dots, \Lambda_{\ell+s}$ . Then we consider the subgroups  $N_l$  generated by  $\dots, \Lambda_{s-1}, \Lambda_s$ , and  $N_r$  generated by  $\Lambda_{s+1}, \Lambda_{s+2}, \dots$ , and, since  $\ell - s > s$ , then  $N = N_l *_{K_s = H_{s+1}} N_r$  with  $n \in N_l$  and  $t^{-\ell}nt^\ell \in N_r$ . Since  $\ell$  is a multiple of  $k$ ,  $t^{-\ell}nt^\ell = n$  and this element is in  $K_s = H_{s+1}$ . By similar arguments (shifting the ‘cutting index’ in the construction of  $N_l, N_r$ ) we obtain that  $n \in K_{s+1} = H_{s+2}, \dots, n \in K_{\ell-s-1} = H_{\ell-s}$ . Hence, the elements  $n' := t^{\ell-s}nt^{s-\ell}, t^{-1}n't = t^{\ell-s-1}nt^{s-\ell+1}, \dots, t^{-\ell+2s+1}n't^{\ell-2s-1} = t^{s+1}nt^{-s-1}$  are in  $H_0 = H$ . Thus, since  $\ell - s > k$ , then  $n', t^{-1}n't, \dots, t^{-k}n't^k \in H$ . Consequently,  $\phi^k(n')$  exists, and  $\phi^k(n') = t^{-k}n't^k = t^{\ell-s-k}nt^{k+s-\ell} = t^{\ell-s}n't^{s-\ell} = n'$ . On the other hand,  $\phi^k$  has no non-trivial fixed point. This is a contradiction.  $\square$

**4.4. Final examples and remarks.** Next, let us look at examples where  $\Gamma_0$  is not cyclic (inspired by Sinclair and Smith, [18]). Let  $\mathbb{Q}$  be the additive group of rational numbers and denote by  $\mathbb{Q}^\times$  the multiplicative group of non-zero rational numbers.

For each positive integer  $n$ , set

$$F_n = \left\{ \frac{p}{q} \cdot 2^{kn} ; p, q \in \mathbb{Z}_{\text{odd}}, k \in \mathbb{Z} \right\} \subset \mathbb{Q}^\times$$

and

$$F_\infty = \left\{ \frac{p}{q} ; p, q \in \mathbb{Z}_{\text{odd}} \right\} \subset \mathbb{Q}^\times.$$

Next, for  $n \in \mathbb{N} \cup \{\infty\}$ , set

$$\Gamma(n) = \left\{ \begin{pmatrix} f & x \\ 0 & 1 \end{pmatrix} ; f \in F_n, x \in \mathbb{Q} \right\}$$

and let  $\Gamma_0(n)$  be the subgroup of diagonal elements of  $\Gamma(n)$ .  $\Gamma(n)$  is an ICC, amenable group. Then the pair  $(\Gamma(n), \Gamma(n)_0)$  satisfies condition (ST) for every  $n$ .

However, if we consider larger matrices, the corresponding pairs of groups do not satisfy condition (ST). Let us fix two positive integers  $m$  and  $n$ , and set

$$\Gamma(m, n) = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & f_1 & 0 \\ 0 & 0 & f_2 \end{pmatrix} ; f_1 \in F_m, f_2 \in F_n, x, y \in \mathbb{Q} \right\},$$

and let  $\Gamma_0(m, n)$  be the corresponding diagonal subgroup. Then

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which belongs to  $\Gamma_0$  for all  $f \in F_m$ . Thus the pair  $(\Gamma(m, n), \Gamma(m, n)_0)$  does not satisfy condition (ST), though it satisfies condition (SS).

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